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ON PERIODIC MOTIONS OF A RIGID BODY IN A CENTRAL NEWTONIAN FIELD

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In the problem of the motion of a rigid body with one fixed point in a central Newtonian force field (in particular, in the de Brun field [1]), The existence of a family of periodic solutions is proved by the Poincaré method of small parameters. It is assumed that the body differs negligibly from a dynamically symmetric one and that its center of gravity is sufficiently close to the fixed point. The proof is carried out by using the techniques of Hamiltonian systems.

We investigate the motion of a rigid body around a fixed point in a Newtonian gravity field, making use for this purpose of the canonical Deprit variables [2] which we introduce as follows. Let $OXYZ$ be a fixed coordinate system with origin at a fixed point O, whose Z-coordinate axis is directed vertically upward, and let $Oxyz$ be a system of axes directed along the principal axes of inertia for point O. Further, let φ, ψ, ϑ be the Euler angles defining the position of the moving system $Oxyz$ relative to the fixed one. We introduce a plane containing point O and perpendicular to kinetic moment G. The position of this plane is given by the longitude h of its nodal line on the OXY plane and its inclination I to this same plane. Finally, we introduce two more Euler angles defining the position of the moving system of axes relative to the plane perpendicular to the kinetic moment: the angle of self-rotation l and the nutation angle b.

As coordinates we now take the angles l, g, h introduced. The canonical momenta associated with them are, respectively, L, G, H, where

$$H = G \cos I, \qquad L = G \cos b$$

The motion of a rigid body with one fixed point in a central Newtonian force field is determined by the Hamiltonian

$$K = T - U$$

$$T = \frac{G^2 - L^2}{2AB}(A \cos^2 l + B \sin^2 l) + \frac{L^2}{2C}$$

$$U = -P(x_c\gamma + y_c\gamma' + z_c\gamma'') - \frac{3P}{2mR}(A\gamma^2 + B\gamma'^2 + C\gamma''^2)$$

Here P is the body's weight, m is its mass, x_c, y_c, z_c are the coordinates of the body's center of inertia in the moving coordinate system, A, B, C are the principal moments of inertia, γ, γ', γ'' are the direction cosines of the fixed point's radius-vector R issuing from the attracting center in the moving system. The direction cosines are expressed in terms of the canonical variables in the following way:

$$\gamma = \frac{1}{G^2}[H\sqrt{G^2 - L^2}\sin l + L\sqrt{G^2 - H^2}\cos g \sin l +$$
$$G\sqrt{G^2 - H^2}\sin g \cos l]$$

$$\gamma' = \frac{1}{G^2}[H\sqrt{G^2 - L^2}\cos l + L\sqrt{G^2 - H^2}\cos g \cos l -$$
$$G\sqrt{G^2 - H^2}\sin g \sin l]$$

$$\gamma'' = \frac{1}{G^2}[HL - \sqrt{(G^2 - L^2)(G^2 - H^2)}\cos g]$$

The subsequent discussions are valid in equal measure for the following two cases. In the first case the center of inertia is sufficiently close to the body's fixed point, while the body itself differs negligibly from a dynamically symmetric one. The other case is characterized by an angular velocity sufficiently large in modulus. In both cases, by a suitable choice of a small parameter we can achieve a partitioning of the Hamiltonian into the two parts needed for the application of Poincaré method of small parameters.

In the first case, as the small parameter we can take the quantity

$$\mu = \max\left\{\sqrt{\frac{A - B}{m}}, \frac{A}{mR}, \frac{B}{mR}, \frac{C}{mR}, x_c, y_c, z_c\right\}$$

We represent the Hamiltonian function in the form

$$K = K_0 + K_1 \tag{1}$$

$$K_0 = \frac{G^2 - L^2}{2A} + \frac{L^2}{2C}, \qquad K_1 = \frac{A - B}{2AB}(G^2 - L^2)\cos^2 l - U$$

Here K_0 is the Hamiltonian of the simplified system of canonical equations of motion defining the generating solution, while K_1 is the perturbing Hamiltonian. We now write out the simplified system of equations of motion

$$\frac{dL}{dt} = 0, \qquad \frac{dG}{dt} = 0, \qquad \frac{dH}{dt} = 0$$

$$\frac{dl}{dt} = \frac{A - C}{AC}L, \qquad \frac{dg}{dt} = \frac{G}{A}, \qquad \frac{dh}{dt} = 0$$

Its general solution is given by the formulas

$$L = L_0 = \text{const}, \quad G = G_0 = \text{const}, \quad H = H_0 = \text{const} \tag{2}$$
$$l = \omega_1 t + \beta_1, \quad g = \omega_2 t + \beta_2, \quad h = \beta_3$$
$$\omega_1 = \frac{A-C}{AC} L_0, \quad \omega_2 = \frac{G_0}{A}$$

Solution (2) is periodic if for any integers k_1 and k_2 we have $k_1\omega_1 = k_2\omega_2$. Here the period of the generating solution

$$\tau = 2\pi k_2/\omega_1 = 2\pi k_1/\omega_2 \tag{3}$$

We now prove the existence of periodic solutions with period (3) of the system of equations with Hamiltonian (1) which coincide with the generating solution when $\mu = 0$. The well-known Poincaré conditions for the existence of periodic solutions for Hamiltonian systems [3] can be simplified if we keep in mind that the equations of motion admit two integrals: the kinetic energy integral and the moment of momentum integral. By virtue of what has been said it is the following small Hessian

$$\frac{D(K'_{0L_0}, K'_{0G_0})}{D(L_0, G_0)} \neq 0 \tag{4}$$

which should be nonzero instead of the Hessian of function K_0.

The second group of periodicity conditions has the form (for simplicity of writing we do not formally delineate the small parameter in K_1)

$$\frac{\partial \bar{K}_1}{\partial \beta_i} = 0, \quad i = 1, 2, 3 \quad \left(\bar{K}_1 = \frac{1}{\tau}\int_0^\tau K_1\, dt\right) \tag{5}$$

Condition (4) is always satisfied when $A \neq C$ since

$$\frac{D(K'_{0L_0}, K'_{0G_0})}{D(L_0, G_0)} = \frac{A-C}{AC} L_0 \neq 0$$

Regarding the second group of conditions (5), the following cases can occur:

$$|k_1| + |k_2| \geqslant 4, \quad |k_1| = 1, \quad |k_2| = 2$$
$$|k_1| = |k_2| = 1, \quad |k_1| = 2, \; |k_2| = 1$$

In the first case the mean value of the Hamiltonian over a period, as shown by computations, does not depend on the constants β_i and, therefore, conditions (5) are fulfilled identically. The last three cases are considered analogously. As an example let us examine the case $k_1 = k_2 = 1$ for which $l - g = \beta_1 - \beta_2$ in the generating solution; this leads to the necessity of additional analysis. Here

$$\bar{K}_1 = \bar{K}_{10}(L_0, G_0, H_0) + \frac{P}{2mG_0^2}\sqrt{G_0^2 - H_0^2}(L_0 - G_0)\,[x_c \sin(\beta_1 - \beta_2) + \tag{6}$$
$$y_c \cos(\beta_1 - \beta_2)] - \frac{3P(A-B)}{16mRG_0^4}(G_0^2 - L_0^2)(G_0 - L_0)^2 \cos 2(\beta_1 - \beta_2)$$

We see from (6) that the third one of the conditions (5) ($i = 3$) is fulfilled identically, while the first two reduce to the one equation

$$\frac{\partial \bar{K}_1}{\partial \beta_1} = -\frac{\partial \bar{K}_1}{\partial \beta_2} = \frac{P}{mG_0^2} \sqrt{G_0^2 - L_0^2}\, [(L_0 + G_0)\, x_c \cos(\beta_1 - \beta_2) - \tag{7}$$

$$(L_0 - G_0)\, y_c \sin(\beta_1 - \beta_2)] + \frac{3P\,(G_0^2 - L_0^2)}{8mR}\,(G_0 - L_0)^2\,(A - B) \sin 2(\beta_1 - \beta_2) = 0$$

It is obvious that condition (7) is fulfilled for a suitable choice of arbitrary constants and, consequently, in the case being considered there also exists a family of periodic solutions (but with a lesser number of arbitrary constants).

We note that in the limit as $R \to \infty$ we arrive at the classical problem of the motion of a rigid body in a homogeneous gravity field, for which periodic solutions are obtained in the first two cases for sufficiently small $A - B$, x_c, y_c, z_c As a more detailed investigation shows, the last two cases lead to periodic solutions inherent only for the de Brun field and the central Newtonian field.

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ON THE ADMISSIBILITY OF APPLICATION OF PRECESSION EQUATIONS OF GYROSCOPIC SYSTEMS

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Precession equations are widely applied in gyroscopic systems. The conditions, under whose fulfillment the application of these equations is in a certain sense legitimate, have been established for linear autonomous systems and for certain special cases of nonlinear systems [1 – 3]. We give below the proof of precession theory for a wide class of nonlinear and nonautonomous systems.

We consider a system under the action of gyroscopic forces depending on a large positive parameter H, resistance forces with total dissipation, and other generalized forces $Q_k(q, t)$ depending on the coordinates q and on time t. Among the generalized forces $Q_k(q, t)$ there can occur potential, position-nonconservative (radial-correction), and other forces depending on the coordinates, perturbing forces depending explicitly on time, inertia forces, etc.

We shall write the equations of motion in the following form [1, 2]: